TOPOLOGY 1
FINAL EXAMINATION

This exam is of **50 marks**. Please **read all the questions carefully** and **do not cheat**. Please feel free to use whatever theorems you have learned in class after stating them clearly. Please **hand in your phones** at the beginning of the class.

1. Let (X, d) be a **complete metric space**. Let $f : X \to X$ be a **contraction** - a continuous map such that

$$d(f(x), f(y)) \leqslant \alpha d(x, y)$$

for some $0 < \alpha < 1$. Show that there is a *unique* point x in X such that $f(x) = x$. (4)

2. Let $\{X_\alpha\}$ be an indexed family of spaces. Show

(1) If $\prod X_\alpha$ is **locally compact**, then each X_α is **locally compact** and further X_α is **compact** for all but finitely many α. (3)

(2) Assuming Tychonoff's theorem, prove the converse. Namely, show that if X_α are **locally compact** and X_α are **compact** for all but finitely many α, then $\prod X_\alpha$ is **locally compact**. (3)

3a. Give an example of a **Hausdorff space** with a **countable basis** which is **not metrizable**. (4)

3b. Let X be **compact** and **Hausdorff**. Show X is **metrizable** if and only if X has a countable basis. (6)

4. Let X be **compact**. Suppose $\{f_n\}$ is a sequence of functions in $\mathcal{C}(X, \mathbb{R}^k)$, the space of continuous functions with values in $\mathbb{R}^k$, such that it is **equicontinuous** and **point-wise bounded** – namely the sequence $\{f_n(x)\}$ is bounded for each $x \in X$. Show that $\{f_n\}$ has a **uniformly convergent subsequence**. (6)

5. Consider the sequence of functions $\{f_n\}$, where $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ is given by

$$f_n(x) = x/n$$

In which of the three topologies on $\mathbb{R}^{\mathbb{R}}$ does it converge?

(1) Uniform Topology (2)

(2) Topology of Compact convergence (2)

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(3) Topology of Point-wise convergence (2)

6. Show that every **locally compact Hausdorff** space is a **Baire** space. (5)

7. If X is a **completely regular space**, show that X is **connected** $\Leftrightarrow$ $\beta(X)$ is **connected**, where $\beta(X)$ is the **Stone-Ĉech compactification**. (5)

8. Let Z be a topological space. A subspace Y is said to be a **retract** of Z if there is a continuous map $r : Z \longrightarrow Y$ such that $r(y) = y$ for all $y \in Y$. Show that

(1) If Z is Hausdorff and Y is a retract of Z then Y is closed in Z. (2)

(2) The set $A = \{(0,0),(1,1)\}$ is not a retract of $\mathbb{R}^2$. (3)

(3) S^1 is a retract of $\mathbb{R}^2 - \{(0,0)\}$. (3)